# Domain-wall and overlap fermions at nonzero quark chemical potential 

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#### Abstract

We have recently given a construction of the overlap Dirac operator at nonzero quark chemical potential. Here, we introduce a quark chemical potential in the domain-wall fermion formalism and show that our earlier result is reproduced if the extent of the fifth dimension is taken to infinity and its lattice spacing is taken to zero. We also extend this result to include a bare quark mass, consider its continuum limit, and prove a number of properties of the overlap operator at nonzero quark chemical potential. In particular, we show that the relation between the anomaly and the index of the overlap operator remains valid.


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## I. INTRODUCTION

The phase diagram of quantum chromodynamics (QCD) as a function of temperature and chemical potential has been the subject of intense studies over many years. It is physically relevant, e.g., for the study of compact stars, for ultra-relativistic heavy-ion collisions, and for the physics of the early universe. The theoretical methods that have been employed to investigate the QCD phase diagram include model calculations, effective theories, perturbative studies at high temperature and density, lattice simulations, and recently also the AdS/CFT correspondence. For an overview of the literature and a summary of the current status, we refer the reader to Ref. [1].

Lattice simulations are the predominant nonperturbative tool to study QCD from first principles. The case of nonzero temperature $T$ can be implemented on the lattice without much effort, and therefore the temperature dependence of many QCD quantities is very well understood, see Ref. [2] for a review. This is not true for the case of nonzero baryon density or, equivalently, quark chemical potential $\mu$. The reason is that at $\mu \neq 0$, the fermion determinant becomes complex so that standard importance sampling methods fail. This is an example of the so-called sign problem, see Ref. [3] for a thorough discussion of this problem in the context of lattice QCD. A number of approaches have been invented to deal with this problem, such as reweighting along the critical line [4], Taylor expansion [5], and analytical continuation from imaginary $\mu[6,7]$. Using these approaches, attempts have been made to determine the transition line $T_{c}(\mu)$ and to locate the critical end-point, see Ref. [8] for a review. While the results are encouraging, the sign problem remains unsolved in principle.

Most of the lattice results for QCD thermodynamics have been obtained with staggered fermions, which reduce the fermion doubling problem and implement a remnant chiral symmetry on the lattice. For simulations with less than four staggered flavors, the so-called rooting problem has been the subject of some debate, see Ref. [9] for a review. We have no intention to enter this debate here. Rather, our aim is to investigate how a quark chem-
ical potential can be implemented in a fermion operator that implements an exact chiral symmetry on the lattice.

In Ref. [10] we showed how this can be done for the overlap operator $[11,12]$. (For earlier work with a similar focus, see Refs. $[13,14]$.) The resulting operator, $D_{\text {ov }}(\mu)$, contains the sign function of a nonhermitian matrix. We also showed that quenched lattice results obtained with this operator agree with analytical predictions from nonhermitian chiral random matrix theory at $\mu \neq 0$ [15-17], see also [18]. Furthermore, the authors of Ref. [19] have shown that our construction of $D_{\text {ov }}(\mu)$ yields the correct energy density for free fermions.

In the present paper, we introduce a quark chemical potential in the domain-wall fermion formalism [20-23], which can be viewed as a particular truncation of the overlap operator. In analogy to the well-known result at $\mu=0$, we show that in the limit in which the extent of the fifth dimension is taken to infinity and its lattice spacing is taken to zero, our earlier result for $D_{\text {ov }}(\mu)$ is reproduced. We also extend this result to include a bare quark mass, consider its continuum limit, and prove several properties of $D_{\mathrm{ov}}(\mu)$, including the relation between the anomaly and the index.

We should remark that at present, the topic we address here may seem to be mainly of theoretical interest since lattice simulations with such an operator, especially at $\mu \neq 0$, are numerically much more expensive than those with staggered fermions. However, as computers and algorithms improve, more and more lattice QCD simulations will be done with overlap and domain-wall fermions. As a first step towards such simulations at $\mu \neq 0$, we have already proposed and tested a new iterative method to compute the sign function of nonhermitian matrices [24].

This paper is organized as follows. In Sec. II we review domain-wall fermions at $\mu=0$, extend the domain-wall action to $\mu \neq 0$, and consider the limit of this action for infinite extent and zero lattice spacing of the fifth dimension. After a short side remark in Sec. III, we discuss the continuum limit of $D_{\mathrm{ov}}(\mu)$ in Sec. IV. In Sec. V we prove a number of properties of $D_{\text {ov }}(\mu)$ that were stated, but not proven, in Ref. [10]. We conclude with a summary and outlook in Sec. VI.

## II. DOMAIN-WALL FERMIONS AT $\mu \neq 0$ AND RELATION TO OVERLAP OPERATOR

## A. Domain-wall fermions at $\mu=0$

We start with the definition of the Wilson Dirac operator in four dimensions, which is given by [25]

$$
\begin{align*}
D_{\mathrm{w}}(\mu)= & (4+M)-\frac{1}{2} \sum_{i=1}^{3}\left(T_{i}^{+}+T_{i}^{-}\right) \\
& -\frac{1}{2}\left(e^{\mu} T_{4}^{+}+e^{-\mu} T_{4}^{-}\right) \tag{2.1}
\end{align*}
$$

with

$$
\begin{equation*}
\left(T_{\nu}^{ \pm}\right)_{x y}=\left(1 \pm \gamma_{\nu}\right) U_{ \pm \nu}(x) \delta_{y, x \pm \hat{\nu}} \tag{2.2}
\end{equation*}
$$

where $M$ is the Wilson mass, the $U \in \mathrm{SU}(3)$ are the lattice gauge fields with $U_{-\nu}(x)=U_{\nu}^{\dagger}(x-\hat{\nu})$, the $\gamma_{\nu}$ are the usual Euclidean Dirac matrices, the Wilson parameter $r$ has been fixed at $r=1$, the 4-d lattice spacing $a$ has been set to unity, and for later convenience we have already included a quark chemical potential $\mu$.

Domain-wall fermions [20-23] are constructed by introducing an additional fifth dimension with lattice spacing $a_{5}$ and extent $L_{s}$. The fermion fields now have an additional index $s=1, \ldots, L_{s}$, while the gauge fields remain four-dimensional and do not depend on $s$. At $\mu=0$, the domain-wall fermion action is given by [22, 23]

$$
\begin{align*}
-S_{5} & =\bar{\psi} D_{5} \psi \\
& =\sum_{s=1}^{L_{s}}\left(\bar{\psi}_{s} A \psi_{s}-\bar{\psi}_{s} P_{R} \psi_{s+1}-\bar{\psi}_{s} P_{L} \psi_{s-1}\right), \tag{2.3}
\end{align*}
$$

where $A=a_{5} D_{\mathrm{w}}(\mu=0)+1$, the chiral projection operators $P_{R}$ and $P_{L}$ are defined by $P_{R / L}=\frac{1}{2}\left(1 \pm \gamma_{5}\right)$, and the fermion fields satisfy the following boundary conditions in the fifth dimension,

$$
\begin{equation*}
P_{R} \psi_{L_{s}+1}=-m P_{R} \psi_{1}, \quad P_{L} \psi_{0}=-m P_{L} \psi_{L_{s}} \tag{2.4}
\end{equation*}
$$

The quantity $m$ is a bare quark mass parameter. In Eq. (2.3), the Wilson mass has to be in the range $-2<$ $M<0$ to obtain renormalizable solutions in the fifth dimension and to avoid the existence of doublers [21, 22].

To take the $L_{s} \rightarrow \infty$ limit, the domain-wall fermion action of Eq. (2.3) is supplemented by a pseudo-fermion action to cancel divergences due to the heavy fermions in the large- $L_{s}$ limit $[23,26-31]$. We use the pseudofermion action of Ref. [30], which is given by Eq. (2.3) with anti-periodic boundary conditions, i.e., $m=1$ in Eq. (2.4), but in which bosonic fields are used instead of the fermionic ones. As the fermion action is only defined up to a constant normalization factor, we choose to multiply the pseudo-fermion action by $1 / 2$.

## B. Domain-wall fermions at $\mu \neq 0$

We define the domain-wall fermion action at $\mu \neq 0$ to be the same action as in Eq. (2.3), except that $D_{\mathrm{w}}(0)$ is replaced by $D_{\mathrm{w}}(\mu)$. With very minor modifications, the arguments of Ref. [21] leading to the bounds on the Wilson mass apply to the case of $\mu \neq 0$ as well, and we again obtain the requirement $-2<M<0$.

## C. $\quad L_{s} \rightarrow \infty$ limit of domain-wall fermions at $\mu \neq 0$

For $\mu=0$, the connection between domain-wall fermions and the overlap operator in the $L_{s} \rightarrow \infty$ limit has been exhibited in a number of earlier works, e.g., Refs. [31-34]. To make the presentation self-contained, we now retrace some of the steps taken in these papers, in particular Refs. [33, 34], with small modifications suitable for our purposes.

The idea is to introduce successive spinor transformations to diagonalize the Dirac operator in the fifth dimension and integrate out the fermion fields. To this end, we start with the transformation

$$
\psi_{s}= \begin{cases}P_{R} \chi_{s}+P_{L} \chi_{s+1} & \text { for } 1 \leq s \leq L_{s}-1  \tag{2.5}\\ P_{R} \chi_{L_{s}}+P_{L} \chi_{1} & \text { for } s=L_{s}\end{cases}
$$

It is straightforward to show that this transformation is orthogonal, with Jacobian equal to 1. Substituting Eq. (2.5) into Eq. (2.3) with boundary conditions (2.4) yields

$$
\begin{align*}
-S_{5} & =\sum_{s=2}^{L_{s}} \bar{\psi}_{s}\left(A P_{R}-P_{L}\right) \chi_{s}+\bar{\psi}_{1}\left(A P_{R}+m P_{L}\right) \chi_{1} \\
& +\sum_{s=2}^{L_{s}} \bar{\psi}_{s-1}\left(A P_{L}-P_{R}\right) \chi_{s}+\bar{\psi}_{L_{s}}\left(A P_{L}+m P_{R}\right) \chi_{1} \tag{2.6}
\end{align*}
$$

To simplify the first term in Eq. (2.6), we introduce the transformation

$$
\begin{equation*}
\bar{\psi}_{s}=\bar{\chi}_{s}\left(A P_{R}-P_{L}\right)^{-1} \tag{2.7}
\end{equation*}
$$

which is diagonal in the fifth dimension. The nontrivial Jacobian of this transformation can be ignored since it is cancelled by the corresponding Jacobian for the pseudofermions. We also define an operator $T$ by

$$
\begin{equation*}
T=-\left(A P_{R}-P_{L}\right)^{-1}\left(A P_{L}-P_{R}\right) \tag{2.8}
\end{equation*}
$$

which is the transfer matrix in the fifth dimension [31] and will be discussed in more detail below. Using Eqs. (2.7) and (2.8), Eq. (2.6) becomes

$$
\begin{align*}
-S_{5} & =\sum_{s=2}^{L_{s}} \bar{\chi}_{s} \chi_{s}+\bar{\chi}_{1}\left(P_{R}-m P_{L}\right) \chi_{1} \\
& -\sum_{s=2}^{L_{s}} \bar{\chi}_{s-1} T \chi_{s}-\bar{\chi}_{L_{s}} T\left(P_{L}-m P_{R}\right) \chi_{1} \tag{2.9}
\end{align*}
$$

where we have used

$$
\begin{array}{ll}
\left(A P_{R}-P_{L}\right)^{-1} A P_{R}=P_{R}, & \left(A P_{R}-P_{L}\right)^{-1} P_{L}=-P_{L} \\
\left(A P_{L}-P_{R}\right)^{-1} A P_{L}=P_{L}, & \left(A P_{L}-P_{R}\right)^{-1} P_{R}=-P_{R} \tag{2.10}
\end{array}
$$

(See below for comments on the invertibility of $A P_{R}-P_{L}$ and $A P_{L}-P_{R}$.) The structure of Eq. (2.9) suggests to transform the $\bar{\chi}_{s}$ for $s>1$ according to

$$
\begin{equation*}
\bar{\eta}_{s}=\bar{\chi}_{s}-\bar{\chi}_{s-1} T \tag{2.11}
\end{equation*}
$$

with inverse transformation

$$
\begin{equation*}
\bar{\chi}_{s}=\bar{\chi}_{1} T^{s-1}+\sum_{i=2}^{s} \bar{\eta}_{i} T^{s-i} \tag{2.12}
\end{equation*}
$$

The $\chi_{s}$ for $s>1$ are transformed according to

$$
\begin{equation*}
\chi_{s}=\eta_{s}+T^{L_{s}+1-s}\left(P_{L}-m P_{R}\right) \chi_{1} \tag{2.13}
\end{equation*}
$$

Both of these transformations have a Jacobian equal to 1. Inserting Eqs. (2.11), (2.12), and (2.13) into Eq. (2.9) leads to

$$
\begin{equation*}
-S_{5}=\sum_{s=2}^{L_{s}} \bar{\eta}_{s} \eta_{s}+\bar{\chi}_{1} D_{4} \chi_{1} \tag{2.14}
\end{equation*}
$$

with

$$
\begin{equation*}
D_{4}=P_{R}-m P_{L}-T^{L_{s}}\left(P_{L}-m P_{R}\right) \tag{2.15}
\end{equation*}
$$

The $\eta_{s}$ and $\bar{\eta}_{s}$ can now be integrated out trivially. Finally, we integrate out $\chi_{1}$ and $\bar{\chi}_{1}$ and obtain, together with the corresponding contribution of the pseudo-fermions,

$$
\begin{equation*}
\frac{\operatorname{det} D_{4}(m)}{\operatorname{det} \frac{1}{2} D_{4}(1)}=\operatorname{det} D_{\mathrm{eff}} \tag{2.16}
\end{equation*}
$$

with an effective 4-d operator $D_{\text {eff }}$ given by

$$
\begin{equation*}
D_{\mathrm{eff}}=(1+m)+(1-m) \gamma^{5} \frac{1-T^{L_{s}}}{1+T^{L_{s}}} \tag{2.17}
\end{equation*}
$$

We now take a closer look at the transfer matrix of Eq. (2.8), which can be rewritten as

$$
\begin{equation*}
T=\left(1+a_{5} H_{\mathrm{w}} P_{R}\right)^{-1}\left(1-a_{5} H_{\mathrm{w}} P_{L}\right) \tag{2.18}
\end{equation*}
$$

with $H_{\mathrm{w}}=\gamma_{5} D_{\mathrm{w}}$. For $\mu=0, D_{\mathrm{w}}$ is $\gamma_{5}$-hermitian, i.e., it satisfies $D_{\mathrm{w}}^{\dagger}=\gamma_{5} D_{\mathrm{w}} \gamma_{5}$, and thus $H_{\mathrm{w}}$ is hermitian. From this it follows that $T$ is also hermitian. The transfer matrix can be related to a 4 -d Hamiltonian $H_{t}$ by writing it in the form

$$
\begin{equation*}
T=\frac{1-a_{5} H_{t}}{1+a_{5} H_{t}} \tag{2.19}
\end{equation*}
$$

with

$$
\begin{equation*}
H_{t}=\left(2+a_{5} H_{\mathrm{w}} \gamma_{5}\right)^{-1} H_{\mathrm{w}}=H_{\mathrm{w}}\left(2+a_{5} \gamma_{5} H_{\mathrm{w}}\right)^{-1} \tag{2.20}
\end{equation*}
$$

For $\mu=0, H_{t}$ is hermitian.
Up to this point, everything went through as in Refs. [33, 34]. We now move on to the case of $\mu \neq 0$, in which $D_{\mathrm{w}}$ ceases to be $\gamma_{5}$-hermitian. Therefore, neither $H_{\mathrm{w}}$ nor $H_{t}$ nor $T$ are hermitian. To obtain the $L_{s} \rightarrow \infty$ limit of Eq. (2.17), we consider the matrix function

$$
\begin{equation*}
f\left(H_{t}\right)=\frac{1-T^{L_{s}}}{1+T^{L_{s}}} \tag{2.21}
\end{equation*}
$$

where $T$ is given by Eq. (2.19) with a nonhermitian matrix $H_{t}$. A function $f$ of an arbitrary complex matrix $C$ can be defined by [35]

$$
\begin{equation*}
f(C)=\frac{1}{2 \pi i} \oint_{\Gamma} d z f(z)[z-C]^{-1} \tag{2.22}
\end{equation*}
$$

where the integral is defined component-wise and $\Gamma$ is a collection of closed contours in $\mathbb{C}$ such that $f$ is analytic inside and on $\Gamma$ and such that $\Gamma$ encloses the spectrum of $C$. We are therefore interested in

$$
\begin{equation*}
\sigma=\frac{1-t^{L_{s}}}{1+t^{L_{s}}} \tag{2.23}
\end{equation*}
$$

where

$$
\begin{equation*}
t=\frac{1-z}{1+z} \tag{2.24}
\end{equation*}
$$

and $z \in \mathbb{C}$ is an eigenvalue of $a_{5} H_{t}$. If $|t|<1(|t|>1)$, $\sigma \rightarrow 1(\sigma \rightarrow-1)$ as $L_{s} \rightarrow \infty$, i.e., we can write $\sigma \rightarrow$ $\varepsilon\left(1-|t|^{2}\right)$, where $\varepsilon$ denotes the sign function. For $z \in \mathbb{R}$, $|t|<1(|t|>1)$ if $z>0(z<0)$, and hence $\sigma=\varepsilon(z)$. Let us now consider $z \in \mathbb{C}$ with $x=\operatorname{Re} z$, for which

$$
\begin{equation*}
1-|t|^{2}=\frac{4 x}{|1+z|^{2}} \tag{2.25}
\end{equation*}
$$

From this expression we obtain immediately

$$
\begin{equation*}
\lim _{L_{s} \rightarrow \infty} \sigma=\varepsilon(x)=\varepsilon(\operatorname{Re} z)=: \varepsilon(z) \tag{2.26}
\end{equation*}
$$

where the last equality defines the sign function of a complex number. This can also be written as

$$
\begin{equation*}
\varepsilon(z)=\frac{z}{\sqrt{z^{2}}} \tag{2.27}
\end{equation*}
$$

with the branch cut of the square root along the negative real axis. We thus obtain

$$
\begin{equation*}
\lim _{L_{s} \rightarrow \infty} D_{\mathrm{eff}}(\mu)=(1+m)+(1-m) \gamma_{5} \varepsilon\left(H_{t}(\mu)\right) \tag{2.28}
\end{equation*}
$$

where the sign function $\varepsilon(C)$ of a nonhermitian matrix $C$ is defined formally by Eq. (2.22) in combination with Eq. (2.26) or (2.27). A simpler form for $\varepsilon(C)$ can
be obtained if $C$ can be diagonalized, i.e., if it can be written in the form $U \Lambda U^{-1}$ with $U \in \operatorname{Gl}(N, \mathbb{C})$ and $\Lambda=\operatorname{diag}\left(\lambda_{1}, \ldots, \lambda_{N}\right)$, where $N$ is the dimension of $C$. It then follows from Eq. (2.22) that

$$
\begin{equation*}
f(C)=U f(\Lambda) U^{-1} \tag{2.29}
\end{equation*}
$$

with

$$
\begin{equation*}
f(\Lambda)=\operatorname{diag}\left(f\left(\lambda_{i}\right)\right) \tag{2.30}
\end{equation*}
$$

so that the matrix sign function can be defined by [36]

$$
\begin{equation*}
\varepsilon(C)=U \varepsilon(\operatorname{Re} \Lambda) U^{-1} \tag{2.31}
\end{equation*}
$$

Note that if $C$ cannot be diagonalized, one can use the Jordan canonical form instead, see Ref. [24] for details.

In the course of the derivation, we have assumed (i) that the operators $A P_{R}-P_{L}$ and $A P_{L}-P_{R}$ are invertible and (ii) that the elements of the diagonal matrix $\Lambda$ are not zero or purely imaginary so that the sign function is well-defined. For any of these assumptions to be violated, the gauge-field would have to be fine-tuned. This happens on a gauge field set of measure zero and can therefore be ignored in practice. The same remark applies to the possibility that $H_{t}(\mu)$ might not be diagonalizable.

## D. $a_{5} \rightarrow 0$ limit

To recover the standard overlap operator, it remains to take the limit $a_{5} \rightarrow 0$ in Eq. (2.28), and therefore in Eq. (2.20), which yields

$$
\begin{align*}
D_{\mathrm{ov}}(\mu) & =\lim _{a_{5} \rightarrow 0} \lim _{L_{s} \rightarrow \infty} D_{\mathrm{eff}}(\mu) \\
& =(1+m)+(1-m) \gamma_{5} \varepsilon\left(H_{\mathrm{w}}(\mu)\right) \tag{2.32}
\end{align*}
$$

For $m=0$, Eq. (2.32) together with Eq. (2.31) agrees with our earlier result [10]. For $m \neq 0$, we see that the quark mass is included in the same way as for $\mu=0$ [31].

## III. SIDE REMARK

Note that for $\mu=0$, there is an expression for the overlap operator that is equivalent to Eq. (2.32), i.e.,

$$
\begin{equation*}
D_{\mathrm{ov}}=(1+m)+(1-m) \frac{D_{\mathrm{w}}}{\sqrt{D_{\mathrm{w}}^{\dagger} D_{\mathrm{w}}}} \tag{3.1}
\end{equation*}
$$

One could be tempted to use this expression for $\mu \neq 0$ as well, with $D_{\mathrm{w}}$ replaced by $D_{\mathrm{w}}(\mu)$. However, the resulting operator is not equivalent to Eq. (2.32) for $\mu \neq 0$ due to the lack of $\gamma_{5}$-hermiticity of $D_{\mathrm{w}}(\mu)$. In particular, it does not satisfy the Ginsparg-Wilson condition and has no exact zero modes at finite lattice spacing. Thus, the expression (3.1) is not suitable for an extension to $\mu \neq 0$.

## IV. CONTINUUM LIMIT OF $\operatorname{Dov}(\mu)$

In this (and only this) section we reintroduce the 4-d lattice spacing $a$ that was set to unity earlier and write the Wilson Dirac operator, with the argument $\mu$ suppressed, in the form [37]

$$
\begin{equation*}
D_{\mathrm{w}}=\frac{1}{a}\left[-(1+s)+a \tilde{D}_{\mathrm{w}}\right] \tag{4.1}
\end{equation*}
$$

where $1+s=-M a$ with $|s|<1$ and $\tilde{D}_{\text {w }}$ is the massless Wilson operator, $\tilde{D}_{\mathrm{w}}=D_{\mathrm{w}}(M=0)$. We therefore have for $H_{\mathrm{w}}=\gamma_{5} D_{\mathrm{w}}$

$$
\begin{align*}
\left(a H_{\mathrm{w}}\right)^{2} & =(1+s)^{2}-a(1+s)\left(\tilde{D}_{\mathrm{w}}+\gamma_{5} \tilde{D}_{\mathrm{w}} \gamma_{5}\right)+\mathcal{O}\left(a^{2}\right) \\
& =(1+s)^{2}+\mathcal{O}\left(a^{2}\right) \tag{4.2}
\end{align*}
$$

where the last step follows from the fact that $\tilde{D}_{\mathrm{w}}$ anticommutes with $\gamma_{5}$ up to terms of order $a$. Using Eqs. (2.22) and (2.27), the matrix sign function can be written as

$$
\begin{equation*}
\varepsilon\left(H_{\mathrm{w}}\right)=\frac{H_{\mathrm{w}}}{\sqrt{H_{\mathrm{w}}^{2}}} \tag{4.3}
\end{equation*}
$$

and we find

$$
\begin{equation*}
\gamma_{5} \varepsilon\left(H_{\mathrm{w}}\right)=-1+\frac{a}{1+s} \tilde{D}_{\mathrm{w}}+\mathcal{O}\left(a^{2}\right) \tag{4.4}
\end{equation*}
$$

Equation (2.32) thus becomes

$$
\begin{align*}
D_{\mathrm{ov}}(\mu) & =\frac{1}{a}\left[(1+m a)+(1-m a) \gamma_{5} \varepsilon\left(H_{\mathrm{w}}\right)\right] \\
& =2 m+\frac{1}{1+s} \tilde{D}_{\mathrm{w}}(\mu)+\mathcal{O}(a) \tag{4.5}
\end{align*}
$$

Since for $a \rightarrow 0$ the Wilson Dirac operator becomes the continuum Dirac operator, this is also true for $D_{\text {ov }}(\mu)$, up to a normalization factor. There was no need in the above derivation to use the $\gamma_{5}$-hermiticity of $\tilde{D}_{\mathrm{w}}$, which is lacking for $\mu \neq 0$. The only input required was the fact that $\tilde{D}_{\mathrm{w}}$ anticommutes with $\gamma_{5}$ in the continuum limit, which holds for $\mu \neq 0$ as well.

## V. PROPERTIES OF $\operatorname{Dov}_{\mathrm{ov}}(\mu)$

In Ref. [10] we stated, but did not prove, a number of properties of $D_{\mathrm{ov}}(\mu)$ at $m=0$. We supply the missing proofs here, assuming $m=0$ throughout this section.

Property 1: For $\mu \neq 0, D_{\mathrm{ov}}(\mu)$ is no longer $\gamma_{5^{-}}$ hermitian but satisfies

$$
\begin{equation*}
\gamma_{5} D_{\mathrm{ov}}(\mu) \gamma_{5}=D_{\mathrm{ov}}^{\dagger}(-\mu) \tag{5.1}
\end{equation*}
$$

instead. From Eq. (2.32) with $m=0$ we obtain

$$
\begin{align*}
\gamma_{5} D_{\mathrm{ov}}(\mu) \gamma_{5} & =1+\varepsilon\left(H_{\mathrm{w}}(\mu)\right) \gamma_{5} \\
D_{\mathrm{ov}}^{\dagger}(-\mu) & =1+\varepsilon^{\dagger}\left(H_{\mathrm{w}}(-\mu)\right) \gamma_{5} \tag{5.2}
\end{align*}
$$

from which Eq. (5.1) follows because of

$$
\begin{align*}
\varepsilon^{\dagger}\left(H_{\mathrm{w}}(-\mu)\right) & =\varepsilon^{\dagger}\left(\gamma_{5} D_{\mathrm{w}}(-\mu)\right)=\varepsilon^{\dagger}\left(D_{\mathrm{w}}(\mu)^{\dagger} \gamma_{5}\right) \\
& =\varepsilon\left(\gamma_{5} D_{\mathrm{w}}(\mu)\right)=\varepsilon\left(H_{\mathrm{w}}(\mu)\right) \tag{5.3}
\end{align*}
$$

In the second step, we used the well-known fact that $D_{\mathrm{w}}$ satisfies Eq. (5.1), which follows from Eq. (2.1) and from $\gamma_{5} T_{\nu}^{ \pm} \gamma_{5}=\left(T_{\nu}^{\mp}\right)^{\dagger}$. In the third step, we used the fact that the sign function satisfies $\varepsilon\left(C^{\dagger}\right)=\varepsilon^{\dagger}(C)$ for any matrix $C$, which follows from Eq. (2.22) with $\varepsilon^{*}(z)=\varepsilon\left(z^{*}\right)$, or from Eq. (2.31). Note that the proof of Property 1 also goes through for $m \neq 0$.

Property 2: $D_{\text {ov }}(\mu)$ satisfies a Ginsparg-Wilson relation [38] of the form

$$
\begin{equation*}
\left\{D, \gamma_{5}\right\}=D \gamma_{5} D \tag{5.4}
\end{equation*}
$$

Setting $m=0$ in Eq. (2.32) and using the shorthand $\varepsilon$ for $\varepsilon\left(H_{\mathrm{w}}(\mu)\right)$, we have

$$
\begin{align*}
\left\{D_{\mathrm{ov}}(\mu), \gamma_{5}\right\} & =2 \gamma_{5}+\varepsilon+\gamma_{5} \varepsilon \gamma_{5} \\
D_{\mathrm{ov}}(\mu) \gamma_{5} D_{\mathrm{ov}}(\mu) & =\left(1+\gamma_{5} \varepsilon\right) \gamma_{5}\left(1+\gamma_{5} \varepsilon\right) \\
& =\gamma_{5}+\gamma_{5} \varepsilon^{2}+\varepsilon+\gamma_{5} \varepsilon \gamma_{5} \tag{5.5}
\end{align*}
$$

Equation (5.4) now follows immediately from $\varepsilon^{2}(C)=1$ for any matrix $C$, which in turn follows from Eq. (2.22) with $\varepsilon^{2}(z)=1$, or from Eq. (2.31).

Property 3: All eigenvalues of $D_{\mathrm{ov}}(\mu)$ that are not equal to 0 or 2 come in pairs $\lambda$ (with eigenvector $\psi$ ) and $\lambda /(\lambda-1)$ (with eigenvector $\gamma_{5} \psi$ ). We first note, with all arguments suppressed, that

$$
\begin{equation*}
D_{\mathrm{ov}} \psi=\left(1+\gamma_{5} \varepsilon\right) \psi=\lambda \psi \tag{5.6}
\end{equation*}
$$

implies

$$
\begin{equation*}
\varepsilon \psi=(\lambda-1) \gamma_{5} \psi \tag{5.7}
\end{equation*}
$$

from which we conclude that

$$
\begin{align*}
D_{\mathrm{ov}}\left(\gamma_{5} \psi\right) & =\gamma_{5} \psi+\gamma_{5} \varepsilon \gamma_{5} \psi=\gamma_{5} \psi+\gamma_{5} \varepsilon \frac{\varepsilon \psi}{\lambda-1} \\
& =\frac{\lambda}{\lambda-1}\left(\gamma_{5} \psi\right) \tag{5.8}
\end{align*}
$$

where we have again used $\varepsilon^{2}=1$. Note that for $\mu=0$, all eigenvalues lie on the circle with radius 1 and center at 1 , in which case $\lambda$ and $\lambda /(\lambda-1)$ are complex conjugates of each other.

Property 4: The mapping $\lambda \rightarrow z=2 \lambda /(2-\lambda)$ projects the pair $\lambda$ and $\lambda /(\lambda-1)$ to a complex conjugate pair $\pm z$. This follows from elementary algebra.

Property 5 makes statements about the eigenvectors of $D_{\mathrm{ov}}(\mu)$ corresponding to eigenvalue 0 or 2 . In these cases, Property 3 implies that $\psi$ and $\gamma_{5} \psi$ are degenerate eigenvectors of $D_{\mathrm{ov}}(\mu)$. This means that $\gamma_{5}$ commutes with $D_{\mathrm{ov}}(\mu)$ in the corresponding degenerate subspace and can thus be diagonalized in this subspace. Because of $\gamma_{5}^{2}=1$ the eigenvalues of $\gamma_{5}$ are $\pm 1$, i.e., the eigenvectors of $D_{\mathrm{ov}}(\mu)$ corresponding to $\lambda=0$ or 2 can be
arranged to have definite chirality. In the following we denote by $n_{\lambda}^{ \pm}$the number of eigenvectors corresponding to $\lambda=0$ or 2 with $\left\langle\gamma_{5}\right\rangle= \pm 1$. Consider now the operator $B=D_{\mathrm{ov}}+\gamma_{5} D_{\mathrm{ov}} \gamma_{5}$, where the argument $\mu$ has been suppressed. It is easily shown that if $\psi_{\lambda}$ is an eigenvector of $D_{\mathrm{ov}}$ with eigenvalue $\lambda$, then $\psi_{\lambda}$ and $\gamma_{5} \psi_{\lambda}$ are degenerate eigenvectors of $B$ with eigenvalue $\lambda^{2} /(\lambda-1)$. For $\lambda \neq 0,2$, we now construct the vectors $\psi_{\lambda}^{ \pm}=\psi_{\lambda} \pm \gamma_{5} \psi_{\lambda}$. According to Property $3, \psi_{\lambda}$ and $\gamma_{5} \psi_{\lambda}$ are linearly independent in this case, and therefore the two vectors $\psi_{\lambda}^{ \pm}$ are nonzero and linearly independent. Moreover, they are also eigenvectors of $\gamma_{5}$ with eigenvalue $\pm 1$, respectively. We now consider $B$ in a basis consisting of the $\psi_{\lambda}^{ \pm}$and of the eigenvectors of $D_{\text {ov }}$ corresponding to $\lambda=0$ and 2 with definite chirality. Since in this basis the operators $\gamma_{5}$ and $B$ are simultaneously diagonal with eigenvalues $\lambda_{i}^{\left(\gamma_{5}\right)}$ and $\lambda_{i}^{(B)}$, respectively, we have

$$
\begin{align*}
& \operatorname{tr}\left(\gamma_{5} B\right)=\sum_{i} \lambda_{i}^{\left(\gamma_{5}\right)} \lambda_{i}^{(B)} \\
& \quad=\frac{1}{2} \sum_{\lambda_{i} \neq 0,2}\left(d_{i}-d_{i}\right) \frac{\lambda_{i}^{2}}{\lambda_{i}-1}+\left(n_{0}^{+}-n_{0}^{-}\right) 0+\left(n_{2}^{+}-n_{2}^{-}\right) 4 \\
& \quad=4\left(n_{2}^{+}-n_{2}^{-}\right) \tag{5.9}
\end{align*}
$$

where in the second line $d_{i}$ is the (accidental) degeneracy of the eigenvalue $\lambda_{i} \neq 0,2$ of $D_{\mathrm{ov}}$ and the factor of $1 / 2$ in front of the sum removes a double counting of eigenvalues. An analogous argument holds for $\tilde{D}_{\text {ov }}=2-D_{\text {ov }}$, which also satisfies the Ginsparg-Wilson relation and for which the roles of $\lambda=0$ and $\lambda=2$ are interchanged, leading to

$$
\begin{equation*}
\operatorname{tr}\left(\gamma_{5} \tilde{B}\right)=4\left(n_{0}^{+}-n_{0}^{-}\right) \tag{5.10}
\end{equation*}
$$

where $\tilde{B}=\tilde{D}_{\text {ov }}+\gamma_{5} \tilde{D}_{\text {ov }} \gamma_{5}$. From $\operatorname{tr}\left(\gamma_{5} \tilde{B}\right)=-\operatorname{tr}\left(\gamma_{5} B\right)$ we conclude that

$$
\begin{equation*}
n_{0}^{+}-n_{0}^{-}=-\left(n_{2}^{+}-n_{2}^{-}\right) \tag{5.11}
\end{equation*}
$$

as stated in Ref. [10]. From $\operatorname{tr}\left(\gamma_{5} B\right)=2 \operatorname{tr}\left(\gamma_{5} D_{\text {ov }}\right)$ we also conclude that

$$
\begin{equation*}
-\operatorname{tr}\left(\gamma_{5} D_{\mathrm{ov}}\right)=2\left(n_{0}^{+}-n_{0}^{-}\right)=2 \operatorname{index}\left(D_{\mathrm{ov}}\right) \tag{5.12}
\end{equation*}
$$

The relation (5.12) between the anomaly and the index of $D_{\text {ov }}$ was already proven for $\mu=0$ in Refs. [29, 39, 40]. Our simple derivation shows that it remains valid at $\mu \neq 0$. (The method introduced in Ref. [40] also works at $\mu \neq 0$ without modifications.) Eq. (5.12) was used in Ref. [10] to explain an observed shift in the number of zero modes of $D_{\mathrm{ov}}(\mu)$ as a function of $\mu$.

Property 6 concerns the normality of $D_{\text {ov }}(\mu)$. From Eqs. (5.1) and (5.4), one easily shows that

$$
\begin{align*}
D_{\mathrm{ov}}(\mu) D_{\mathrm{ov}}^{\dagger}(-\mu) & =D_{\mathrm{ov}}(\mu)+D_{\mathrm{ov}}^{\dagger}(-\mu)  \tag{5.13}\\
& =D_{\mathrm{ov}}^{\dagger}(-\mu) D_{\mathrm{ov}}(\mu)
\end{align*}
$$

This means that for $\mu=0, D_{\text {ov }}$ is a normal operator, whereas for $\mu \neq 0$, we cannot conclude anything from

Eq. (5.13) about the normality of $D_{\mathrm{ov}}(\mu)$. This suggests that for $\mu \neq 0, D_{\mathrm{ov}}(\mu)$ is not a normal operator (at least generically). This expectation is confirmed numerically. It is interesting to note that the operator (5.13) is equal to the operator $B$ we defined in the proof of Property 5 .

## VI. SUMMARY

We have extended the domain-wall formalism to nonzero quark chemical potential and have shown that in the limit in which $L_{s} \rightarrow \infty$ and $a_{5} \rightarrow 0$ we obtain an expression for the overlap Dirac operator that is identical to our earlier result [10]. We have also included a bare quark mass, considered the continuum limit, and proven a number of analytical properties of this operator.

In actual lattice simulations, the use of Eq. (2.32) will be hindered by two problems. The first is the infamous sign problem that plagues lattice QCD at $\mu \neq 0$. We have nothing new to say about this problem here. The second problem is that not much is known about efficient numerical computations of the sign function of a nonher-
mitian matrix. As remarked earlier, we have started to address the second problem in Ref. [24].

Work is in progress in several directions. First, we will continue our algorithmic developments to compute the sign function of nonhermitian matrices, with particular emphasis on novel deflation schemes. Second, we are currently testing the predictions of nonhermitian random matrix theory also for the unquenched theory, which can be done by reweighting on the small lattices that we have studied so far. We are also studying the average phase factor of the fermion determinant, for which some analytical predictions exist in the epsilon-regime of QCD [41, 42].

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